Natural Language Processing

Logistic Regression

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Components of a probabilistic machine learning classifier

Given m input/output pairs $(x^{(i)}, y^{(i)})$:

1. A **feature representation** for the input. For each input observation $x^{(i)}$, a vector of features $[x_1, x_2, \ldots, x_n]$. Feature j for input $x^{(i)}$ is x_j, more completely $x_1^{(i)}$, or sometimes $f_j(x)$.

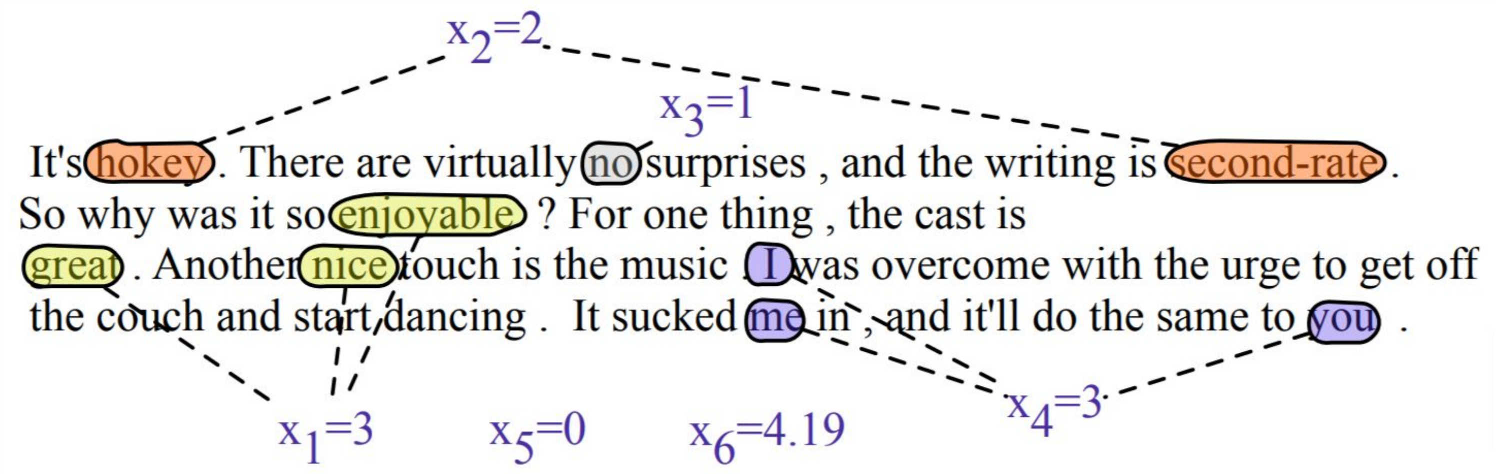
2. A **classification function** that computes $\hat{y}$ the estimated class, via $p(y|x)$, like the **sigmoid** functions

3. An **objective function** for learning, like **cross-entropy loss**

4. An algorithm for **optimizing** the objective function: **stochastic gradient descent**

Sentiment example: does y=1 or y=0?

It's hokey . There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable ? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .



Var	Definition	Value
x_1	count(positive lexicon) $\in$ doc)	3
x_2	count(negative lexicon) $\in$ doc)	2
x_3	$\begin{cases} 1 & \text{if “no”} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$	1
x_4	count(1st and 2nd pronouns $\in$ doc)	3
x_5	$\begin{cases} 1 & \text{if “!”} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$	0
x_6	log(word count of doc)	$\ln(66) = 4.19$

Classifying sentiment for input x

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Suppose w = [2.5, -5.0, -1.2, 0.5, 2.0, 0.7]

b = 0.1

An objective function or loss

We want loss to be:

- smaller if the model estimate $\hat{y}$ is close to correct
- bigger if model is confused

Deriving cross-entropy loss for a single observation x

Goal: maximize probability of the correct label $p(y|x)$

Since there are only 2 discrete outcomes (0 or 1) we can express the probability $p(y|x)$ from our classifier (the thing we want to maximize) as

$$p(y|x) = \hat{y}^{y}(1-\hat{y})^{1-y}$$

Noting:

if $y=1$, this simplifies to $\hat{y}$

if $y=0$, this simplifies to $1 - \hat{y}$

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Maximize: $p(y|x) = \hat{y}^y (1-\hat{y})^{1-y}$

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Now take the log of both sides (mathematically handy)

Maximize:
$$\begin{aligned}\log p(y|x) &= \log \left[\hat{y}^y (1-\hat{y})^{1-y}\right] \\ &= y \log \hat{y} + (1-y) \log (1-\hat{y})\end{aligned}$$

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Whatever values maximize log p(y|x) will also maximize p(y|x)

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Now flip sign to turn this into a loss: something to minimize

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$$L_{\mathrm{CE}}(\hat{y},y) = -\log p(y|x) = -[y\log\hat{y} + (1-y)\log(1-\hat{y})]$$

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Plug in definition of $\hat{y} = \sigma(w \cdot x + b)$

$$L_{\mathrm{CE}}(\hat{y},y) = -\left[y\log\sigma(\mathbf{w}\cdot\mathbf{x}+b) + (1-y)\log\left(1-\sigma(\mathbf{w}\cdot\mathbf{x}+b)\right)\right]$$

Learning components

A loss function:

- **cross-entropy loss**

An optimization algorithm:

- **stochastic gradient descent**

Stochastic Gradient Descent

- Stochastic Gradient Descent algorithm
  - is used to optimize the weights
  - for logistic regression
  - also for neural networks

Our goal: minimize the loss

Let's make explicit that the loss function is parameterized by weights $\Theta=(w,b)$

- And we'll represent $\hat{y}$ as $f(x; \theta)$ to make the dependence on θ more obvious

We want the weights that minimize the loss, averaged over all examples:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \frac{1}{m} \sum_{i=1}^{m} L_{CE}(f(x^{(i)};\theta), y^{(i)})$$

$$L_{CE}(\hat{y}, y)$$

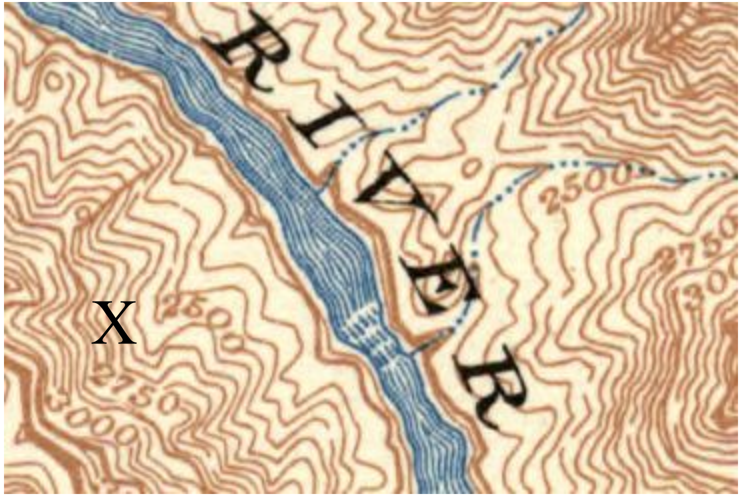
Intuition of gradient descent

How do I get to the bottom of this river canyon?

Look around me 360◦

Find the direction of steepest slope down

Go that way



Our goal: minimize the loss

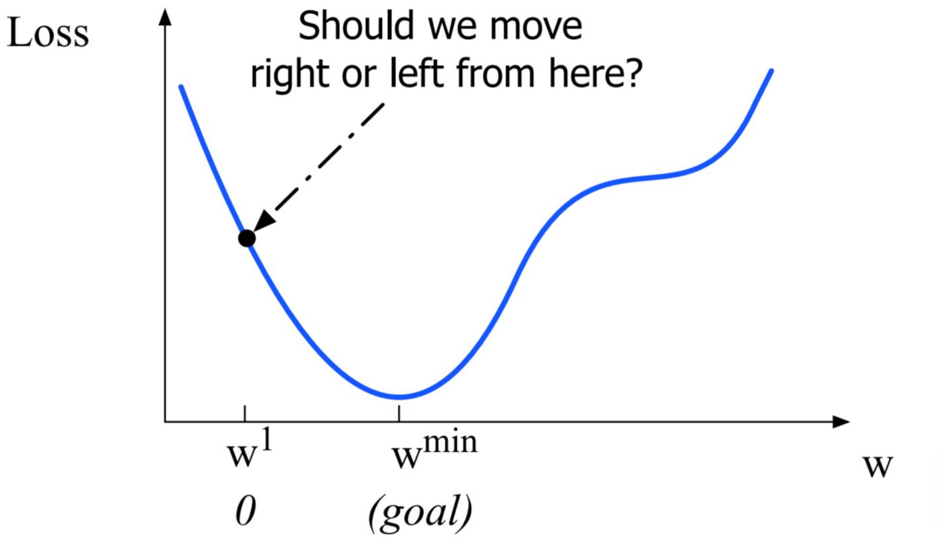
For logistic regression, loss function is **convex**

- A convex function has just one minimum
- Gradient descent starting from any point is guaranteed to find the minimum
  - (Loss for neural networks is non-convex)

Let's first visualize for a single scalar w

Q: Given current w, should we make it bigger or smaller?

A: Move w in the reverse direction from the slope of the function



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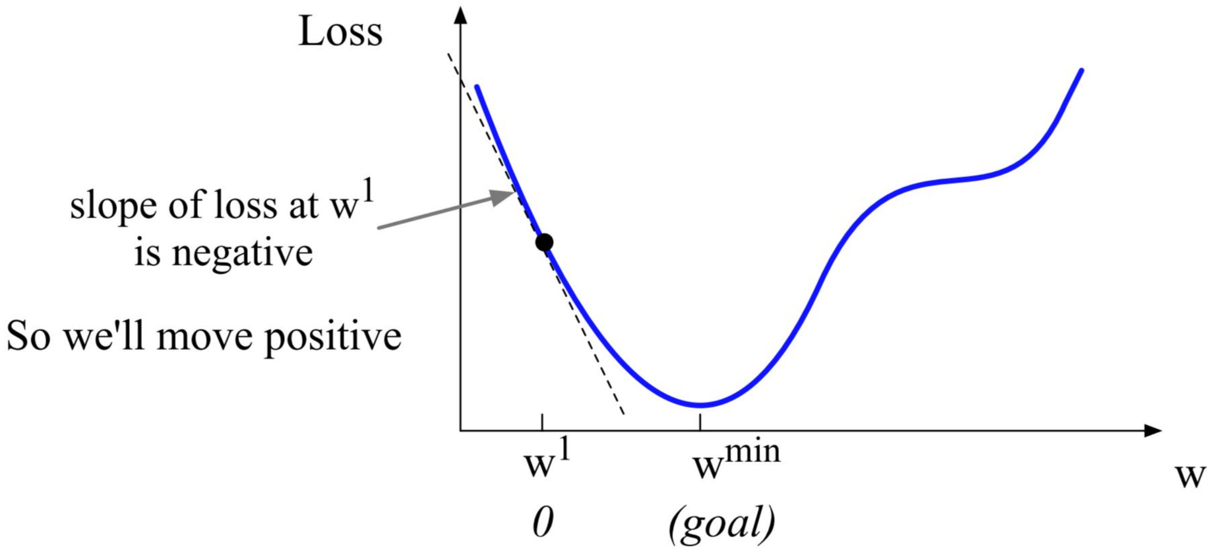

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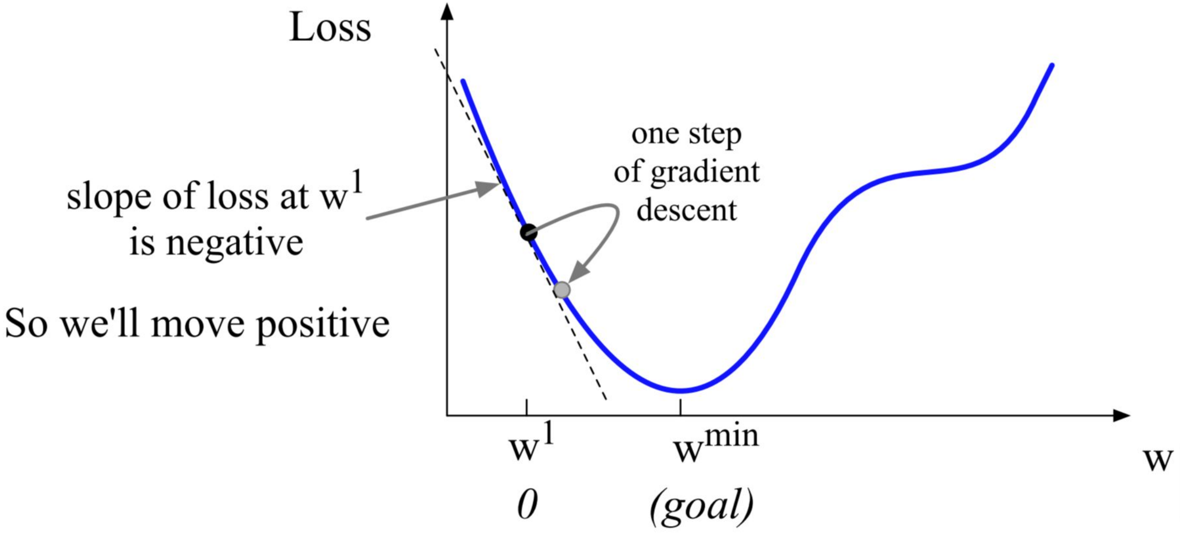


Gradients

The **gradient** of a function of many variables is a vector pointing in the direction of the greatest increase in a function.

Gradient Descent: Find the gradient of the loss function at the current point and move in the **opposite** direction.

How much do we move in that direction?

- The value of the gradient (slope in our example) $\frac{d}{dw}L(f(x;w),y)$
  - weighted by a learning rate η

- Higher learning rate means move w faster

$$w^{t+1} = w^t - \eta \frac{d}{dw}L(f(x;w),y)$$

Now let's consider N dimensions

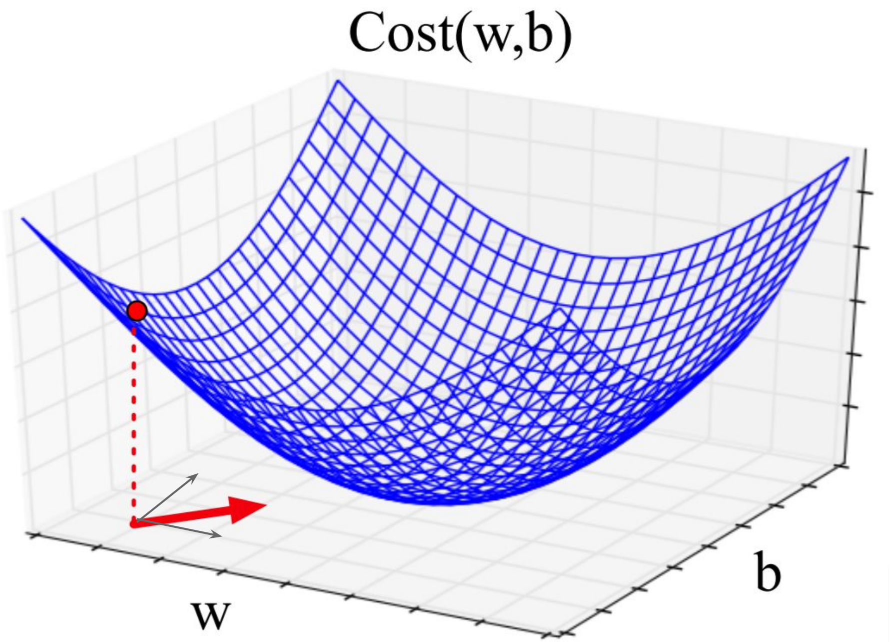
We want to know where in the N-dimensional space (of the N parameters that make up θ) we should move.

The gradient is just such a vector; it expresses the directional components of the sharpest slope along each of the N dimensions.

Imagine 2 dimensions, w and b

Visualizing the gradient vector at the red point

It has two dimensions shown in the x-y plane



Real gradients

Are much longer; lots and lots of weights

For each dimension w_i the gradient component i tells us the slope with respect to that variable.

- "How much would a small change in w_i influence the total loss function L?"
- We express the slope as a partial derivative ∂ of the loss ∂w_i $\quad \frac{\partial}{\partial w_i}$

The gradient is then defined as a vector of these partials.

The gradient

We'll represent $\hat{y}$ as $f(x; \theta)$ to make the dependence on θ more obvious:

$$\nabla_\theta L(f(x;\theta), y)) = \begin{bmatrix} \frac{\partial}{\partial w_1} L(f(x;\theta), y) \\ \frac{\partial}{\partial w_2} L(f(x;\theta), y) \\ \vdots \\ \frac{\partial}{\partial w_n} L(f(x;\theta), y) \end{bmatrix}$$

The final equation for updating θ based on the gradient is thus:

$$\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta), y)$$

What are these partial derivatives for logistic regression?

The loss function

$$L_{CE}(\hat{y}, y) = -[y \log \sigma(w \cdot x + b) + (1 - y) \log (1 - \sigma(w \cdot x + b))]$$

The elegant derivative of this function (see Section 5.10 for the derivation)

$$\frac{\partial L_{CE}(\hat{y}, y)}{\partial w_j} = [\sigma(w \cdot x + b) - y]x_j$$

$$= (\hat{y} - y)\mathbf{x}_j$$

function STOCHASTIC GRADIENT DESCENT($L(), f(), x, y$) **returns** θ

where: L is the loss function
f is a function parameterized by θ
x is the set of training inputs $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$
y is the set of training outputs (labels) $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$

$\theta \leftarrow 0$

repeat til done

For each training tuple $(x^{(i)}, y^{(i)})$ (in random order)

1. Optional (for reporting): # How are we doing on this tuple?
   Compute $\hat{y}^{(i)} = f(x^{(i)}; \theta)$ # What is our estimated output $\hat{y}$?
   Compute the loss $L(\hat{y}^{(i)}, y^{(i)})$ # How far off is $\hat{y}^{(i)}$) from the true output $y^{(i)}$?
2. $g \leftarrow \nabla_\theta L(f(x^{(i)}; \theta), y^{(i)})$ # How should we move θ to maximize loss?
3. $\theta \leftarrow \theta - \eta\, g$ # Go the other way instead

return θ

Hyperparameters

The learning rate η is a **hyperparameter**

- too high: the learner will take big steps and overshoot
- too low: the learner will take too long

Hyperparameters:

- Briefly, a special kind of parameter for an ML model
- Instead of being learned by algorithm from supervision (like regular parameters), they are chosen by algorithm designer.

Mini-batch training

Stochastic gradient descent chooses a single random example at a time.

That can result in choppy movements

More common to compute gradient over batches of training instances.

Batch training: entire dataset

Mini-batch training: m examples (512, or 1024)

Overfitting

A model that perfectly match the training data has a problem.

It will also **overfit** to the data, modeling noise

- A random word that perfectly predicts y (it happens to only occur in one class) will get a very high weight.
- Failing to generalize to a test set without this word.

A good model should be able to **generalize**

Regularization

A solution for overfitting

Add a **regularization** term R(θ) to the loss function (for now written as maximizing logprob rather than minimizing loss)

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{m} \log P(y^{(i)}|x^{(i)}) - \alpha R(\theta)$$

Idea: choose an R(θ) that penalizes large weights

- fitting the data well with lots of big weights not as good as fitting the data a little less well, with small weights

L2 regularization (ridge regression)

The sum of the squares of the weights

$$R(\theta) = ||\theta||_2^2 = \sum_{j=1}^{n} \theta_j^2$$

L2 regularized objective function:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{m} \log P(y^{(i)}|x^{(i)}) \right] - \alpha \sum_{j=1}^{n} \theta_j^2$$

L1 regularization (=lasso regression)

The sum of the (absolute value of the) weights

$$R(\theta) \;=\; ||\theta||_1 = \sum_{i=1}^{n} |\theta_i|$$

L1 regularized objective function:

$$\hat{\theta} \;=\; \underset{\theta}{\operatorname{argmax}} \left[\sum_{1=i}^{m} \log P(y^{(i)}|x^{(i)}) \right] - \alpha \sum_{j=1}^{n} |\theta_j|$$

Multinomial Logistic Regression

Often we need more than 2 classes

- Positive/negative/neutral
- Parts of speech (noun, verb, adjective, adverb, preposition, etc.)
- Classify emergency SMSs into different actionable classes

If >2 classes we use **multinomial logistic regression**

= Softmax regression

= Multinomial logit

= (defunct names : Maximum entropy modeling or MaxEnt)

So "logistic regression" will just mean binary (2 output classes)

Multinomial Logistic Regression

The probability of everything must still sum to 1

P(positive|doc) + P(negative|doc) + P(neutral|doc) = 1

Need a generalization of the sigmoid called the **softmax**

- Takes a vector $z = [z_1, z_2, ..., z_k]$ of k arbitrary values
- Outputs a probability distribution
- each value in the range $[0,1]$
- all the values summing to 1

We'll discuss it more when we talk about neural networks

The **softmax** function

- Turns a vector $z = [z_1, z_2, ..., z_k]$ of k arbitrary values into probabilities

$$\text{softmax}(z_i) = \frac{\exp(z_i)}{\sum_{j=1}^{k} \exp(z_j)} \quad 1 \leq i \leq k$$

- The denominator $\sum_{i=1}^{k} e^{z_i}$ is used to normalize all the values into probabilities

$$\text{softmax}(z) = \left[\frac{\exp(z_1)}{\sum_{i=1}^{k} \exp(z_i)}, \frac{\exp(z_2)}{\sum_{i=1}^{k} \exp(z_i)}, \ldots, \frac{\exp(z_k)}{\sum_{i=1}^{k} \exp(z_i)} \right]$$

softmax: a generalization of sigmoid

- For a vector z of dimensionality k, the softmax is:

$$\text{softmax}(z) = \left[\frac{\exp(z_1)}{\sum_{i=1}^{k}\exp(z_i)}, \frac{\exp(z_2)}{\sum_{i=1}^{k}\exp(z_i)}, \ldots, \frac{\exp(z_k)}{\sum_{i=1}^{k}\exp(z_i)}\right]$$

$$\text{softmax}(z_i) = \frac{\exp(z_i)}{\sum_{j=1}^{k}\exp(z_j)} \quad 1 \leq i \leq k$$

Example:

$$z = [0.6, 1.1, -1.5, 1.2, 3.2, -1.1]$$

$$\text{softmax}(z) = [0.055, 0.090, 0.006, 0.099, 0.74, 0.010]$$

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Next class:

- Language models